MULTIPLE EXISTENCE OF PERIODIC SOLUTIONS FOR LIENARD SYSTEM(Nonlinear Analysis and Mathematical Economics)

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MULTIPLE EXISTENCE OF PERIODIC SOLUTIONS FOR LIENARD SYSTEM

N. Hirano (平野　載倫)
Department of Mathematics
Yokohama National University
Yokohama, JAPAN

W. S. Kim*
Department of Mathmatics
Dong-A University
Pusan, Republic of KOREA

Abstract. The multiple existence of periodic solutions of nonlinear Lienard system is treated. The proof is based on the theory of topological degree and monotone operators.

1. Introduction. The purpose in this present paper is to consider the multiple existence of solutions to the periodic problem of the Lienard system of the form :

$$x'' - \frac{d}{dt}G(x) + f(t,x) = e \tag{E}$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{B}$$

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where $e \in R^N$, and $G : R^N \to R^N$ and $f : R \times R^N \to R^N$ are continuous function. More precisely, we discuss the existence of a constant $R_0 \in R^N$ with $e > R_0$ for which the problem (P) has at least 2^N solutions.

This type of result, so called an Ambrosetti-Prodi type result(briefly APT result), has been initiated by Ambrosetti-Prodi [1] in 1972 in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. A notable discussion for APT results for periodic solutions has been done by Fabry. Mawhin and Nkashama [3] for second order ordinary differential equations with one-side coercive nonlinearity and they particulized their results to Lienard equations having a coercive nonlinearity. A similar result for periodic solutions of the first order ordinary differential equations has been made by Mawhin [5]. In their work, the proofs made use of the upper-lower solution method and degree theory. For APT results to the higher order (≥ 3) ordinary differential equations having a coercive nonlinearity, we refer to read Ding and Mawhin [2]. They used degree theory and Lyapunov-Schmidt argument and they imposed an unilateral Lipschitz condition on the nonlinear term when the order is even.

We refer also to read Ramos and Sanchez [6], and Ramos [7] for APT results of periodic solutions for higher order (≥ 3) ordinary differential equations with a coercive nonlinear term. They treated APT results when the nonlinear term has an one-sided growth restriction. They made use of variational method and degree theory.

For our result, we impose the following conditions:

$f : R \times R^N \to R^N$ is a continuous function of the form

$$f(t,x) = g(x) + h(t,x)$$

where $g : R^N \to R^N$ is a continuous functions of the form

$$g(x) = (g_1(x_1), \cdots, g_N(x_N)) \quad \text{for all} \quad x = (x_1, \cdots, x_N) \tag{1.1}$$

and

$$\lim_{|x| \to \infty} g_k(x) = \infty, \quad k = 1, \cdots, N. \tag{1.2}$$

$h : R \times R^N \to R^N$ is a continuous mapping and satisfies

$$\sup\{|h(t,x)| : (t,x) \in R \times R^N\} < M \quad \text{for some} \quad M > 0. \tag{1.3}$$

$G \in C^1(R^N, R^N)$ satisfies that there exists $c > 0$ and $d > 0$ with $d < 1$ such that

$$|G(x) - G(y)| < d|x - y| \quad \text{for all} \quad x, y \in R^N \tag{1.4}$$

and

$$(G'(x)y, y) > c|y|^2 \quad \text{for all} \quad x, y \in R^N \tag{1.5}$$

where $G'(x)$ is the Frechet derivative of G.

Remark. If $G'(x)$ is independent of x, we do not need the conditions (1.4) and (1.5). We need that $A = G'(x)$ is a strongly positive definite matrix with $\|A\| < 1$.

Theorem. Assume that G and f satisfies (1.1)-(1.5). Then there exists $R_0 > 0$ such that for each $e \in R^N$ with $e_k > R_0$ for all $1 < k < N$, the problem (P) possesses at least 2^N solutions.

2. Proof of Theorem. We first introduce notations we need. We denote by $\|.\|$ and $(\, , \,)$ the norm and inner product, respectively, of the space $L^2((0,2\pi), R^N)$, C_p^r denotes the Banach space of 2π-periodic functions $x : R \to R^N$ of class C^r. The norm of C_p^0 is defined by $\|x\|_\infty = \sup\{|x(t)| : t \in [0,2\pi]\}$ for $x \in C_p^0$. We put $C_p^\infty = \cap_{r=1}^\infty C_p^r$.

We denote H the subspace of C_p^1 defined by

$$H = \{x \in C^1(R, R^N) : x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}.$$

For each $e \in H$, we write $e = \bar{e} + \tilde{e}$ with

$$\bar{e} = \frac{1}{2\pi}\int_0^{2\pi} e(t)dt, \quad \int_0^{2\pi} \tilde{e}(t)dt = 0.$$

The subspace $\widetilde{H}$ of H is defined by

$$\widetilde{H} = \{x \in H : \bar{x} = 0\}.$$

Then H has the decompition $H = \widetilde{H} \oplus R^N$. The projections from H onto $\widetilde{H}$ and R^N are denoted by $\widetilde{P}$ and $\bar{P}$, respectively. Then

$$x = \tilde{x} + \bar{x} = \widetilde{P}x + \bar{P}x \quad \text{for all} \quad x \in H.$$

Moreover, we denote by $\bar{P}_i$ the i-th component of $\bar{P}$. That is

$$\bar{P}x = (\bar{P}_1 x, \cdots, \bar{P}_N x) \quad \text{for} \quad x \in H.$$

The identity mappings on $\widetilde{H}$ and R^N are denoted by $\widetilde{I}$ and $\bar{I} = (\bar{I}_1, \cdots, \bar{I}_N)$, respectively. For each $r > 0$, we denote by $J(r)$ open interval $(0, r)$, $-J(r)$ stands for the interval $(-r, 0)$. We denote by $\bar{J}(r)$ closed interval $[0, r]$, $-\bar{J}(r)$ stands for the interval $[-r, 0]$.

Let E be a subspace of $L^2((0, 2\pi), R^N)$ defined by

$$E = \{x \in L^2((0, 2\pi), R^N) : \bar{x} = 0\}.$$

We set

$$V = \{x \in E : x' \in L^2((0, 2\pi), R^N)\}.$$

Then V is a Hilbert space with the norm

$$\|x\|_V^2 = \|x\|^2 + \|x'\|^2 \quad \text{for} \quad x \in V,$$

and continuously embedded in E (we write $V \hookrightarrow E$). We denote by V^* the dual space of V. Then $V \hookrightarrow E \hookrightarrow V^*$. For each $\bar{x} \in R^N$, we define a mapping $L_{\bar{x}}$ from V into its dual space V^* by

$$\langle L_{\bar{x}} x, y \rangle = \langle x', y' \rangle + \langle G'(x + \bar{x}) x', y \rangle \quad \text{for all} \quad x, y \in V.$$

If we define a subset H_0 of H by $H_0 = \{x \in H : x'' \in L^2((0, 2\pi), R^N)\}$. Then the problem (P) is equivalent to the abstract equation $-L_{\bar{x}} \widetilde{x} = e - f$ in H_0.

Now we have

Lemma 1. For each $\bar{x} \in R^N, L_{\bar{x}} : V \to V^*$ is a continuous and strongly monotone mapping.

Proof. It is obvious from the definition that $L_{\bar{x}}$ is continuous. Let $x, y \in V$. Then we have

$$\begin{aligned}\langle L_{\bar{x}}x - L_{\bar{x}}y, x-y\rangle =& \|x'-y'\|^2 + \langle G'(x+\bar{x})x' - G'(y+\bar{x})y', x-y\rangle \\ =& \|x'-y'\|^2 - \langle G(x+\bar{x}) - G(y+\bar{x}), x'-y'\rangle \\ \geq& \|x'-y'\|^2 - d\|x-y\|\|x'-y'\|.\end{aligned}$$

Then nothing that $\|x\| \leq \|x'\|$ for $x \in V$, we find that

$$\langle L_{\bar{x}}x - L_{\bar{x}}y, x-y\rangle \geq (1-d)\|x'-y'\|^2 \geq (1-d)\|x-y\|^2 \quad \text{for all} \quad x, y \in V.$$

This completes the proof.

It follows from Lemma 1 that $E \subset R(L_{\bar{x}})$ and $L_{\bar{x}}$ is injective. Hence $L_{\bar{x}}^{-1} : E \to V \subset E$ is well defined. Again from Lemma 1, we see that the operator $f \mapsto L_{\bar{x}}^{-1}f$ from E into V is bounded. Since V is compactly imbedded in E, we find that $L_{\bar{x}}^{-1}$ is a compact operator.

Lemma 2. If we define $\widetilde{H}_0 = \{x \in H_0 : \bar{x} = 0\}$, then $\widetilde{H}_0 = L_{\bar{x}}^{-1}(E)$.

Proof. It is clear that $\widetilde{H}_0 \subseteq L_{\bar{x}}^{-1}(E)$. Let $f \in E$ and suppose that $L_{\bar{x}}^{-1}(f) = x$. Then there exists a sequence $\{x_n\}$ in C_p^∞ such that $x_n \to x$ in V. By the continuity of $L_{\bar{x}}$, we have $L_{\bar{x}}x_n \to L_{\bar{x}}x$ in $L^2((0,2\pi), R^N)$. If we put $L_{\bar{x}}x_n = f_n$, then clearly $f_n \to f$ in $L^2((0,2\pi), R^N)$, $x_n'' - G'(\bar{x}+x_n)x_n' =$

$-f_n$ and $x_n'(0) = x_n'(2\pi)$. Since $x_n \to x$ in $V, x_n \to x$ in C_p^0 and $x_n' \to x'$ in $L^2((0,2\pi), R^N)$. Hence $x_n'' \to G'(\bar{x}+x)x' - f$ in $L^2((0,2\pi), R^N)$ and thus

$$\int_{t_0}^{t} x_n''(s)ds \to \int_{t_0}^{t} [G'(\bar{x}+x(s))x'(s) - f(s)]ds$$

for all $t, t_0 \in [0, 2\pi]$.

Since $x_n' \to x'$ a.e. in $[0,2\pi]$, for $t_0 \in [0,2\pi]$ such that $x_n'(t_0) \to x'(t_0)$, we have

$$x'(t) - x'(t_0) = \int_{t_0}^{t} [G'(\bar{x}+x(s))x'(s) - f(s)]ds$$

a.e. in $[0, 2\pi]$.

Hence $x'' - G'(\bar{x}+x)x' = -f$ a.e. on $[0,2\pi]$ and so $x'' \in L^2((0,2\pi), R^N)$. Since $x_n'' \to x''$ in $L^2((0,2\pi), R^N)$ and $\bar{x_n'} = \bar{x'} = 0$, by the Sobolev inequality, $x_n' \to x'$ in C_p^0. Hence $x \in C^1(R, R^N)$ and $x'(0) = x'(2\pi)$. Therefore $x \in \widetilde{H}_0$ and thus $\widetilde{H}_0 = L_{\bar{x}}^{-1}(E)$.

Lemma 3. There exists $M_0 > 0$ such that for any solution x of (P),

$$\|\widetilde{x}\|_\infty < M_0. \tag{2.1}$$

Proof. Let $x \in H$ be a solution of (P). We multiply (E) by x' and integrate over $[0, 2\pi]$. Then nothing that x satisfies (B), we find that

$$c\|x'\|^2 \leq \int_0^{2\pi} |h(t,x)||x'|dt \leq 2\pi M\|x'\|.$$

Therefore

$$\|x'\| \leq 2\pi M/c.$$

By the Sobolev inequality, the assertion follows.

Here we put

$$W = \{\widetilde{x} \in \widetilde{H} : \|\widetilde{x}\|_\infty \leq M_0\}.$$

We define a family $\{T_s : s \in [0,1]\}$ of mappings from $W \times R^N$ into $\widetilde{H} \times R^N$ by

$$T_s \begin{pmatrix} \widetilde{x} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \widetilde{T}(\widetilde{x}, \bar{x}) \\ \bar{T}(\widetilde{x}, \bar{x}) \end{pmatrix} = \begin{pmatrix} L_{\bar{x}}^{-1}(\widetilde{P}(g(x) + sh(t,x))) \\ \bar{P}(g(x) + sh(t,x) - e + \bar{x}) \end{pmatrix}$$

where $x = \widetilde{x} + \bar{x}$. If $(\widetilde{x}, \bar{x})$ is a fixed point of T_s for some $s \in [0.1]$, Then

$$\widetilde{P}(x'' - \frac{d}{dt}G(x) + g(x) + sh(t,x)) = 0 \tag{2.2}$$

and

$$\bar{P}(g(x) + sh(t,x)) = e. \tag{2.3}$$

It is easy to see that (2.2) and (2.3) implise that $x = \widetilde{x} + \bar{x}$ is a solution of the problem

$$x'' - \frac{d}{dt}G(x) + g(x) + sh(t,x) = e. \tag{P_s}$$

If $s = 1, x$ is a solution of (P).

We will show that if we choose R_0 sufficiently large, the mapping T_0 possesses at least 2^N fixed points for each $e \in R^N$, with $e_i > R_0, i = 1, \cdots, N$.

Here we choose a positive number R_0 so large that

$$(2.4)\qquad (\sup\{g_k(t) : |t| \le M_0\} + M) < R_0 \quad \text{for all} \quad 1 \le k \le N.$$

It then follows that for each $\widetilde{x} \in \widetilde{H}$ with $\|\widetilde{x}\|_\infty \le M_0$,

$$(2.5)\qquad \bar{P}_k(g_k(\widetilde{x}_k) + sh_k(t, \widetilde{x})) < R_0$$

for all $s \in [0,1]$ and $k = 1, \cdots, N$. Now we fix $e \in R^N$ such that

$$e_i > R_1 \quad \text{for all} \quad i = 1, \cdots, N.$$

We next choose a positive number R_1 such that $R_1 > R_0$ and

$$\inf\{g_k(t) : |t| > R_1 - M_0\} > e_k + M \quad \text{for all} \quad k = 1, \cdots, N.$$

This implies that

$$(2.6)\qquad \bar{P}_k(g_k(\widetilde{x}_k + \bar{x}_k) + sh_k(t, x)) > e_k$$

for all $x \in H$ with $\widetilde{x} \in W$ and $|\bar{x}_k| > R_1$. We set $J(R_1) = (0, R_1)$. We also set

$$K = \{(i_1, \cdots, i_N) : i_k = \pm 1 \quad \text{for} \quad 1 \le k \le N\}.$$

Then K contains 2^N elements.
Now let

$$D_0 = W \times \prod_{k=1}^{N} i_k \bar{J}(R_1).$$

If $x \in H$ is a solution of (P) in D_0, then

$$\|x\|_\infty \leq R_1 + \frac{\pi}{\sqrt{3}} M/c.$$

Multiply (E) by x'' and integrate over $[0, 2\pi]$, then

$$\int_0^{2\pi} (x''(t))^2 dt - \int_0^{2\pi} G'(x(t))x'(t)x''(t)dt$$

$$+ \int_0^{2\pi} g(x(t))x''(t)dt + \int_0^{2\pi} h(t, x(t))x''(t)dt = 0.$$

Since $G \in C^1(R^N, R^N), g : R^N \to R^N$ is continuous and $|h(t,x)| \leq M$ for all $(t,x) \in R \times R^N$, we have

$$\|x''\| \leq M_1' \quad \text{for some} \quad M_1' > 0$$

where M_1' depends only on c, R_1, G, g and h.
Consequently, there exists a constant $M_1 > 0$ such that

$$\|x'\|_\infty < M_1$$

for any possible solution of (P) lying in D_0.
Define D by

$$D = [W^0 \times \prod_{k=1}^{N} i_k J(R_1)] \cap \{x \in H : \|x'\|_\infty < M_1\}$$

where $W^0 = \{\tilde{x} \in \widetilde{H} : \|\tilde{x}\|_\infty < M_0\}$.

Then we have the following :

Lemma 4. For each $(i_1, \cdots, i_N) \in K$,

$$\deg(I - T_0, D, 0) = 1.$$

Proof. Let $(i_1, \cdots, i_N) \in K$. We define a homotopy of compact mappings

$$F_s(x) = (\widetilde{F}_s(x), F_{s,1}(x), \cdots, F_{s,N}(x)), \quad 0 \le s \le 1,$$

on D by

$$\widetilde{F}_s(x) = (1-s)L_{\bar{x}}^{-1}(\widetilde{P}(g(x)))$$

and

$$F_{s,k}(x) = (1-s)(\bar{x}_k - i_k(g_k(x_k)) - e_k) - sz_k, \quad 1 \le k \le N.$$

Here $z = (z_1, \cdots, z_N)$ is a fixed vector such that $z_k = -i_k\delta$ for some sufficent small positive number δ.

From the definition of F_s, we have that $F_0 = T_0$ and

$$F_1(x) = (0, i_1\delta, \cdots, i_N\delta) \quad \text{for all} \quad x \in V. \tag{2.7}$$

Now let $s \in [0,1]$ and $x \in D$ be a fixed point of F_s. Then x satisfies

$$L_{\bar{x}}\tilde{x} - (1-s)\widetilde{P}g(x) = 0. \tag{2.8}$$

and

$$(1-s)(i_k(g_k(x_k) - e_k)) + s(\bar{x}_k + z_k) = 0. \tag{2.9}$$

Then we can see from Lemma 3 that $\widetilde{x} \notin \partial W^0$. On the other hand, if $\bar{x}_k = 0$ for some $1 \leq k \leq N$, then by (2.5) we have

$$(1-s)(i_k(g_k(\widetilde{x}_k) - e_k)) - si_k\delta \neq 0.$$

This contradicts to (2.9). That is $\bar{x}_k \neq 0$ for any $1 \leq k \leq N$. Suppose next that $\bar{x}_k = i_k R_1$ for some $1 \leq k \leq N$. Then by (2.6),

$$(1-s)(i_k(g_k(x_k) - e_k)) + s(i_k R_1 - i_k\delta) \neq 0. \tag{2.10}$$

Then form the argument above, we obtain that $x \notin \partial D$. Therefore from the invariance of degree under homotopy, we have that

$$\deg(I - T_0, D, 0) = \deg(I - F_0, D, 0) = \deg(I - F_1, D, 0).$$

We can see from (2.7) that $\deg(I - F_1, D, 0) = 1$ Therefore the assertion follows.

Proof of Theorem We can see from (2.4),(2,2) and (2.6) that

$$T_s x \neq x \quad \text{for} \quad x \in \partial(D) \quad \text{and} \quad 0 \leq s \leq 1.$$

Then by the homotopy invariance of degree, we have from Lemma 4 that

$$\deg(I - T_1, D) = 1 \tag{2.11}$$

for any $(i_1, \cdots, i_N) \in K$. In fact, if (2.11) holds each $(i_1, \cdots, i_N) \in K$, the problem (P) has a solution in D. Therefore (P) possesses at least 2^N solutions.

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